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Some Asymptotic Behavior Results for Initial Value Problems: An Application of Invariant Imbedding*

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I. INTRODUCTION

The method of invariant imbedding has been successfully applied to a number of problems in various areas of physics and applied mathematics. In [1] G. M. Wing applied the technique to the study of the asymptotic behavior of solutions to certain initial value problems. In particular, the following two problems were considered:

$$u''(t) + u(t) = f(t)\,u(t), \qquad u(x) = \cos\theta, \qquad u'(x) = \sin\theta \tag{1.1}$$

$$u''(t) - u(t) = f(t)\,u(t), \qquad u(x) = \cos\theta, \qquad u'(x) = \sin\theta, \tag{1.2}$$

where f is continuous and $\int^{\infty} |f| < \infty$.

Clearly for fixed x and θ (1.1) and (1.2) each has a unique solution for all $t \geqslant x$. But by considering x and θ as variables each of the above two problems describes a two-parameter family of problems with solutions denoted by $u(t; x, \theta)$. It was shown in [1] that the solutions to (1.1) and (1.2) can be expressed

$$u(t; x, \theta) = A(x, \theta)\cos\{t - \tilde{\psi}(x, \theta)\} + o(1) \tag{1.3}$$

$$u(t; x, \theta) = e^{t-x}\{\mathscr{A}(x, \theta) + o(1)\}, \tag{1.4}$$

respectively, as $t \to \infty$. Expressions for A, $\tilde{\psi}$, and $\mathscr{A}$ were derived which involve only f; i.e., the expressions are independent of the solutions, $u(t; x, \theta)$, themselves. Clearly, knowledge of the quantities A, $\tilde{\psi}$, $\mathscr{A}$ describe quite adequately the asymptotic behavior of the solutions.

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More recently W. A. Beyer successfully applied the above approach to a problem arising in scattering theory, [2]. Expressions analogous to A and $\tilde{\psi}$ in (1.3) were derived for the equation

$$w'' + \left(1 - \frac{2}{t}\eta + \frac{L(L+1)}{t^2}\right) w = f(t)\, w.$$

Here we generalize the above ideas to include a fairly large class of second-order equations. In the next section we *initiate* the imbedding approach for the equation

$$u'' + c(t)\, u = f(t)\, u,$$

where the only assumption on c is continuity. Several lemmas are established for this general equation which are useful in the subsequent sections.

In Sections III-V we discuss the three standard forms for $c(t)$; namely, $c(t) = 1 + g(t)$, $c(t) = -(1 + g(t))$ and $c(t) = 0$ where $g(t)$ is appropriately "small" as $t \to \infty$. In these cases sufficient information concerning two independent solutions to the respective differential equation is known and we are able to derive expressions analogous to A, $\tilde{\psi}$, or $\mathscr{A}$ above.

Section VI is devoted to applications of the results to certain parameter studies and the last section suggests more difficult problems to which the technique can possibly be applied in the future.

II. Some Preliminary Results

In this section we initiate the invariant imbedding approach on a rather general second-order initial value problem and establish some lemmas which will be used later.

We consider the following problem

$$u'' + c(t)\, u = f(t)\, u, \qquad u(x) = \cos\theta, \qquad u'(x) = \sin\theta; \tag{2.1a}$$

$$\text{where } f \text{ and } c \text{ are continuous for } t \geqslant x \geqslant x_0, \tag{2.1b}$$

and

$$\int_{x_0}^{\infty} |f| < \infty. \tag{2.1c}$$

Let $u_1(t)$ and $u_2(t)$ denote two solutions to

$$u'' + c(t)\, u = 0$$

such that the Wronskian $[u_1(t), u_2(t)] \equiv 1$. Let the solutions to (2.1) be denoted by $u(t; x, \theta)$ for $t \geqslant x$. Using the variation of constants formula we can express $u(t; x, \theta)$ as follows

$$u(t; x, \theta) = u_1(t) \left\{ \cos \theta u_2'(x) - \sin \theta u_2(x) - \int_x^t u_2(s) f(s)\, u(s; x, \theta)\, ds \right\}$$

$$+ u_2(t) \left\{ \sin \theta u_1(x) - \cos \theta u_1'(x) + \int_x^t u_1(s) f(s)\, u(s; x, \theta)\, ds \right\}. \qquad (2.2)$$

We will make several applications of the following lemma, a proof of which may be found in [3].

LEMMA 2.1 (Gronwall's inequality). *Let g and h be continuous and non-negative for $t \geqslant t_0$. If, for some $b \geqslant 0$,*

$$g(t) \leqslant b + \int_{t_0}^t g(s)\, h(s)\, ds$$

then

$$g(t) \leqslant b \exp \left[\int_{t_0}^t h(s)\, ds \right].$$

If the solutions to $u'' + cu = 0$ have bounded solutions only, using Lemma 2.1 we now show that the solutions to (2.1) are *uniformly* bounded.

LEMMA 2.2. *Suppose u_1, u_1', u_2, and u_2' are bounded for $t \geqslant x_0$. Then for some M,*

$$| u(t; x, \theta) | \leqslant M$$

for $t \geqslant x$, for $x \geqslant x_0$, and for all θ.

PROOF. By using (2.2)

$$| u(t; x, \theta) | \leqslant M_1 + M_2 \int_x^t | f |\, | u(s; x, \theta) |\, ds.$$

By Gronwall's inequality it follows that

$$| u(t; x, \theta) | \leqslant M_1 \exp \left[M_2 \int_x^t | f | \right] \leqslant M_1 \exp \left[M_2 \int_{x_0}^\infty | f | \right];$$

which proves the lemma.

The following result is important for our work.

LEMMA 2.3. *If $u(t; x, \theta)$ denotes the solutions to problem (2.1) then, for small Δ,*

$$u(x + \Delta; x, \theta) = \eta(x, \theta)\, u(x + \Delta; x + \Delta, \theta') = \eta(x, \theta) \cos \theta'$$
$$u'(x + \Delta; x, \theta) = \eta(x, \theta)\, u'(x + \Delta; x + \Delta, \theta') = \eta(x, \theta) \sin \theta', \tag{2.3}$$

where

$$\eta(x, \theta) = 1 + \Delta \sin \theta \cos \theta[f(x) - c(x) + 1] + o(\Delta)$$
$$\theta'(x, \theta) = \theta + \Delta\{\cos^2 \theta[f(x) - c(x) + 1] - 1\} + o(\Delta) \tag{2.4}$$

as $\Delta \to 0$.

PROOF. Using the mean-value theorem we can write

$$\begin{aligned} u(x + \Delta; x, \theta) &= u(x; x, \theta) + u'(\tilde{x}; x, \theta)\, \Delta \\ &= \cos \theta + \Delta \sin \theta + o(\Delta) \\ u'(x + \Delta; x, \theta) &= u'(x; x, \theta) + u''(\bar{x}; x, \theta)\, \Delta \\ &= \sin \theta + \Delta[f(x) - c(x)] \cos \theta + o(\Delta) \end{aligned}$$

as $\Delta \to 0$. We define

$$\begin{aligned} \eta(x, \theta) &\equiv + \{[u(x + \Delta; x, \theta)]^2 + [u'(x + \Delta; x, \theta)]^2\}^{1/2} \\ &= \{1 + 2\Delta \sin \theta \cos \theta[f(x) - c(x) + 1] + o(\Delta)\}^{1/2} \\ &= 1 + \Delta \sin \theta \cos \theta[f(x) - c(x) + 1] + o(\Delta) \end{aligned}$$

as $\Delta \to 0$, which is the desired expression for $\eta(x, \theta)$.

By using this expression for η and (2.3),

$$\begin{aligned} \sin \theta' &= \frac{u'(x + \Delta; x, \theta)}{\eta(x, \theta)} \\ &= \sin \theta + \Delta \cos \theta\{\cos^2 \theta[f(x) - c(x) + 1] - 1\} + o(\Delta) \end{aligned}$$

as $\Delta \to 0$. Moreover, by using Taylor's formula

$$\sin \theta' = \sin \theta + (\theta' - \theta) \cos \theta - \tfrac{1}{2} (\theta' - \theta)^2 \sin \tilde{\theta}.$$

Using these two expressions for $\sin \theta'$ to solve for θ' we obtain the expression in (2.4) and prove the lemma.

A convenient interpretation of (2.3) is as follows. Assume the solution $u(t; x, \theta)$ is available on the small interval $[x, x + \Delta]$ (or $[x + \Delta, x]$ if $\Delta < 0$). The values $u(x + \Delta; x, \theta)$ and $u'(x + \Delta; x, \theta)$ are normalized by dividing by $\eta(x, \theta)$ and the normalized quantities, $\cos \theta'$ and $\sin \theta'$, are used as initial values for a new problem with solution $u(t; x + \Delta, \theta')$. Since the differential equation is linear homogeneous it follows that

$$u(t; x, \theta) = \eta(x, \theta)\, u(t; x + \Delta, \theta') \tag{2.5}$$

for all $t > x$. This will be important in what follows.

If, as in Lemma 2.2, we add the hypothesis that the solutions to $u'' + cu = 0$ are all bounded we can proceed a little further in the treatment of problem (2.1). Referring to Lemma 2.2 and to (2.1c) it is clear that if $u_1(t)$ and $u_2(t)$, and hence $u(t; x, \theta)$, are bounded functions of t that the two integrals in (2.2) converge as $t \to \infty$. This allows us to write

$$\begin{aligned} u(t; x, \theta) = u_1(t) &\left\{\cos \theta u_2'(x) - \sin \theta u_2(x) - \int_x^\infty u_2 f u(s; x, \theta)\, ds\right\} \\ &+ u_2(t) \left\{\sin \theta u_1(x) - \cos \theta u_1'(x) + \int_x^\infty u_1 f u(s; x, \theta)\, ds\right\} \\ &+ \int_t^\infty [u_1(t)\, u_2(s) - u_2(t)\, u_1(s)]\, f(s)\, u(s; x, \theta)\, ds \\ \equiv\; & A(x, \theta)\, u_1(t) + B(x, \theta)\, u_2(t) + \xi(t; x, \theta) \end{aligned} \tag{2.6}$$

and it is clear that $\xi(t; x, \theta) \to 0$ as $t \to \infty$.

If, for fixed x and θ, the values of $A(x, \theta)$ and $B(x, \theta)$ were known and if u_1 and u_2 (or their asymptotic behavior) were known then the asymptotic behavior of $u(t; x, \theta)$ would be determined. The expressions for $A(x, \theta)$ and $B(x, \theta)$ given in (2.6) will be of some use to us but they certainly do *not* provide a feasible algorithm for computing A and B since u_1, u_2 and $u(t; x, \theta)$ are in general unknown. However (2.6) allows us to prove that A and B have the "smoothness" which we will need later.

LEMMA 2.4. *$A(x, \theta)$ and $B(x, \theta)$, defined in (2.6), are continuously differentiable in x and θ for $x > x_0$ and for all θ.*

PROOF. It follows almost directly from standard theorems concerning dependence upon initial values (e.g. see Chapter 1 of [4]) that $\partial u/\partial x$ and $\partial u/\partial \theta$ are continuous in t, x, and θ.

Differentiating $u(t; x, \theta)$, expressed by (2.2), with respect to x and θ and application of Gronwall's inequality shows that for some M_1 and M_2

$$\left| \frac{\partial u}{\partial x}(t; x, \theta) \right| \leqslant M_1 + M_2 f(x); \qquad \left| \frac{\partial u}{\partial \theta}(t; x, \theta) \right| \leqslant M_1$$

for $t \geqslant x \geqslant x_0$ and for all θ. Note that f may be unbounded but since it is continuous will be bounded on finite intervals.

By differentiating $A(x, \theta)$, given in (2.6), we have

$$\frac{\partial A}{\partial x} = \cos \theta u_2''(x) - \sin \theta u_2'(x) - \int_x^\infty u_2 f \frac{\partial u}{\partial x}(s; x, \theta)\, ds + u_2(x) f(x) \cos \theta. \tag{2.7}$$

For any finite x-interval it is easily seen that the integral in (2.7) converges uniformly in x and θ. This justifies the differentiation under the integral sign and establishes the continuity of $\partial A/\partial x$.

In the same manner it can be shown that $\partial A/\partial \theta$, $\partial B/\partial x$, and $\partial B/\partial \theta$ are continuous and this proves the lemma.

In order to complete the work we must known the asymptotic behavior of independent solutions to the "unperturbed" equation

$$u'' + c(t)\, u = 0.$$

We now examine some of the standard forms of this equation for which this information is known and derive expressions for the desired coefficients; i.e., A and B above or analogous expressions.

III. The Equation $u'' + (1 + g(t))\, u = f(t)\, u$

We now consider the following problem.

$$u'' + (1 + g(t))\, u = f(t)\, u, \qquad u(x) = \cos \theta, \qquad u'(x) = \sin \theta; \tag{3.1a}$$

$$\text{where } f \text{ is continuous and } \int_{x_0}^\infty |f| < \infty, \tag{3.1b}$$

and

$$g \text{ is absolutely continuous, } \int_{x_0}^\infty |g'| < \infty, \tag{3.1c}$$
$$\text{and } g(t) \to 0 \text{ as } t \to \infty.$$

From well-known theorems (see [3]) we are guaranteed solutions to the equation $u'' + (1 + g)\, u = 0$ such that

$$\begin{aligned} u_1(t) &= \cos \alpha(t) + o(1), \qquad & u_1'(t) &= -\sin \alpha(t) + o(1) \\ u_2(t) &= \sin \alpha(t) + o(1), \qquad & u_2'(t) &= \cos \alpha(t) + o(1) \end{aligned} \tag{3.2}$$

as $t \to \infty$, where

$$\alpha(t) = \int_b^t \sqrt{1 + g}\, ds.$$

Now b can be arbitrarily chosen, $b \geqslant x$, but to avoid the possibility that

$u_1(t)$ and $u_2(t)$ are complex valued we pick b large enough to guarantee that $1 + g(s) > 0$ for $s > b$. (Actually, solutions of the form (3.2) are guaranteed for $u'' + (1 + g)\,u = fu$; for the sake of uniformity we choose to consider fu as the "perturbation" term throughout.)

Referring to (2.6) we can express the solutions to (3.1) as

$$\begin{aligned} u(t; x, \theta) &= A(x, \theta)\, u_1(t) + B(x, \theta)\, u_2(t) + \tilde{\xi}(t; x, \theta) \\ &= A(x, \theta) \cos \alpha(t) + B(x, \theta) \sin \alpha(t) + \xi(t; x, \theta), \end{aligned}$$

where $\xi(t; x, \theta) \to 0$ as $t \to \infty$ and

$$\begin{aligned} A(x, \theta) &= \cos \theta[\cos \alpha(x) + o(1)] - \sin \theta[\sin \alpha(x) + o(1)] \\ &\qquad - \int_x^\infty u_2 fu(s; x, \theta)\, ds \\ B(x, \theta) &= \sin \theta[\cos \alpha(x) + o(1)] + \cos \theta[\sin \alpha(x) + o(1)] \\ &\qquad + \int_x^\infty u_1 fu(s; x, \theta)\, ds \end{aligned}$$

as $x \to \infty$. It follows that

$$\begin{aligned} A(x, \theta) &= \cos \theta \cos \alpha(x) - \sin \theta \sin \alpha(x) + o(1) = \cos (\theta + \alpha(x)) + o(1) \\ B(x, \theta) &= \sin \theta \cos \alpha(x) + \cos \theta \sin \alpha(x) + o(1) = \sin (\theta + \alpha(x)) + o(1) \end{aligned}$$

as $x \to \infty$; moreover, by Lemma 2.2 $u(s; x, \theta)$ is uniformly bounded and hence the above convergence is *uniform in* θ.

We make a final change in our expression for $u(t; x, \theta)$. Define

$$\begin{aligned} M(x, \theta) &= +\sqrt{A^2 + B^2} \\ \tilde{\psi}(x, \theta) &= \arctan \frac{B}{A} \\ \psi(x, \theta) &= \tilde{\psi}(x, \theta) - \theta - \alpha(x), \end{aligned} \tag{3.3}$$

where the branch of $\tilde{\psi}$ and the possible complication when $A(x, \theta) = 0$ is discussed below. Using (3.3) we can now write

$$\begin{aligned} u(t; x, \theta) &= M(x, \theta) \cos \{\alpha(t) - \tilde{\psi}(x, \theta)\} + \xi(t; x, \theta) \\ &= M(x, \theta) \cos \{\alpha(t) - \psi(x, \theta) - \theta - \alpha(x)\} + \xi(t; x, \theta). \end{aligned} \tag{3.4}$$

Our goal is to solve for M and ψ. To that end we establish the following lemma.

LEMMA 3.1. *The solutions to problem (3.1) are given by (3.4), where:*

(i) *For* $x > x_0$ $M(x, \theta)$ *is positive and continuously differentiable in* x *and* θ. *Also* $\lim_{x\to\infty} M(x, \theta) = 1$ *uniformly in* θ.

(ii) $\xi(t; x, \theta) \to 0$ *as* $t \to \infty$ *uniformly in* θ *and* x.

(iii) $\psi(x, \theta)$ *is continuously differentiable in* x *and* θ *for* $x > x_0$ *and the branch of* ψ *can be selected such that* $\lim_{x\to\infty} \psi(x, \theta) = 0$ *uniformly in* θ.

PROOF. It follows from standard results (see [4], Chapter 3) that $u(t; x, \theta) \to 0$ iff $u(t; x, \theta) \equiv 0$. Since the initial values do not allow $u(t; x, \theta) \equiv 0$ this implies that $M(x, \theta) > 0$. Moreover, since A and B are continuously differentiable by Lemma 2.4, it follows that M is also continuously differentiable for $x > x_0$.

It was pointed out above that

$$A(x, \theta) \to \cos(\theta + \alpha(x)) \qquad \text{and} \qquad B(x, \theta) \to \sin(\theta + \alpha(x))$$

as $x \to \infty$ uniformly in θ. Hence $M^2 = A^2 + B^2 \to 1$ uniformly in θ. This proves (i).

Clearly (ii) follows immediately from the definition of $\xi(t; x, \theta)$; see (2.6) and Lemma 2.2.

Earlier we showed that $A(x, \theta) \to \cos(\theta + \alpha(x))$, $B(x, \theta) \to \sin(\theta + \alpha(x))$, and $M(x, \theta) \to 1$ as $x \to \infty$ uniformly in θ. Hence, since $\sin\tilde{\psi} = B/M$ and $\cos\tilde{\psi} = A/M$,

$$\begin{aligned}
\sin\psi &= \sin(\tilde{\psi} - \theta - \alpha(x)) = \sin\tilde{\psi}\cos(\theta + \alpha(x)) - \cos\tilde{\psi}\sin(\theta + \alpha(x)) \\
&= [\sin(\theta + \alpha(x)) + o(1)]\cos(\theta + \alpha(x)) \\
&\qquad\qquad - [\cos(\theta + \alpha(x)) + o(1)]\sin(\theta + \alpha(x)) \\
&= o(1)
\end{aligned}$$

as $x \to \infty$ and clearly the convergence is uniform in θ. It follows that $\psi(x, \theta) \to m\pi$ for some integer m and uniformly so in θ. Finally, we *choose* $m = 0$; i.e., we select this branch of the arctangent function of (3.3) and have the desired result

$$\lim_{x\to\infty} \psi(x, \theta) = 0 \text{ uniformly in } \theta.$$

It remains to guarantee the smoothness of $\psi(x, \theta)$. By Lemma 2.4 and well-known properties of the arctangent function, $\tilde{\psi}(x, \theta) = \arctan(B/A)$ (and therefore $\psi(x, \theta)$) is continuously differentiable as long as $A(x, \theta) \neq 0$. We indicate how $\tilde{\psi}(x, \theta)$ can be defined at and near such points so that it has the required smoothness.

Let (x_1, θ_1) be a point such that $A(x_1, \theta_1) = 0$. Then $B(x_1, \theta_1) \neq 0$ and by continuity $B(x, \theta) \neq 0$ for some neighborhood, N, of (x_1, θ_1). Assume further that for some point (x_2, θ_2) in N the value $\tilde{\psi}(x_2, \theta_2)$ has already been determined. We extend the definition of $\tilde{\psi}$ throughout N by

$$\tilde{\psi}(x, \theta) \equiv \arccos \frac{A(x, \theta)}{M(x, \theta)},$$

where the branch of the arccosine is determined by $\tilde{\psi}(x_2, \theta_2)$. Since $B \neq 0$ in N then $A^2 = M^2 - B^2 < M^2$ and it follows that $\tilde{\psi}$ is single-valued and continuously differentiable in N. This completes the proof of the lemma.

LEMMA 3.2. *For $x > x_0$ $M(x, \theta)$ and $\psi(x, \theta)$ satisfy:*

(i) $$\frac{\partial M}{\partial x} + \frac{\partial M}{\partial \theta}[\cos^2 \theta(f - g) - 1] = -\sin \theta \cos \theta(f - g)\, M$$

$$\lim_{x \to \infty} M(x, \theta) = 1 \quad \textit{uniformly in } \theta.$$

(ii) $$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial \theta}[\cos^2 \theta(f - g) - 1] = -[\cos^2 \theta(f - g) - 1] - \sqrt{1 + g}$$

$$\lim_{x \to \infty} \psi(x, \theta) = 0 \quad \textit{uniformly in } \theta.$$

PROOF. As was pointed out in (2.5)

$$u(t; x, \theta) = \eta(x, \theta)\, u(t; x + \Delta, \theta')$$

for $t > x$. It follows from this that

$$M(x, \theta) = \eta M(x + \Delta, \theta')$$

$$\psi(x, \theta) + \theta + \alpha(x) = \psi(x + \Delta, \theta') + \theta' + \alpha(x + \Delta). \tag{3.5}$$

Using (2.4) we have

$$\eta(x, \theta) = 1 + \Delta \sin \theta \cos \theta[f(x) - g(x)] + o(\Delta);$$

moreover,

$$M(x + \Delta, \theta') = M(x + \Delta, \theta) + (\theta' - \theta)\left[\frac{\partial M}{\partial \theta}(x, \theta) + o(1)\right]$$

$$= M(x + \Delta, \theta) + \Delta[\cos^2 \theta(f - g) - 1]\frac{\partial M}{\partial \theta}(x, \theta) + o(\Delta)$$

as $\Delta \to 0$. Hence, by using (3.5),

$$[M(x + \Delta, \theta) - M(x, \theta)] + \Delta\, [\cos^2 \theta\, (f - g) - 1]\, \frac{\partial M}{\partial \theta}(x, \theta)$$
$$= -\Delta \sin \theta \cos \theta[f - g]\, M(x + \Delta, \theta) + o(\Delta).$$

Finally, dividing through by Δ and letting $\Delta \to 0$ we have

$$\frac{\partial M}{\partial x} + \frac{\partial M}{\partial \theta}[\cos^2\theta(f-g) - 1] = -\sin\theta\cos\theta(f-g)\,M,$$

and since we have already shown that $M(x, \theta) \to 1$ as $x \to \infty$, uniformly in θ, this establishes (i).

In the same manner the required partial differential equation for ψ, given in (ii) above, is obtained. This proves the lemma.

We use the method of characteristics to solve the above boundary value problems as follows. Let (x_1, θ_1) be an arbitrary point where $x_1 \geqslant x_0$. We have the following system to consider.

$$\begin{aligned} x'(s) &= 1; \qquad x(x_1) = x_1 \qquad (\text{hence } x(s) \equiv s) \\ \theta'(s) &= \cos^2\theta(s)\,(f(s) - g(s)) - 1; \qquad \theta(x_1) = \theta_1 \\ M'(s) &= -\sin\theta(s)\cos\theta(s)\,[f(s) - g(s)]\,M(s); \qquad M(x_1) = M(x_1, \theta_1). \end{aligned}$$

We seek an expression for $M(x_1, \theta_1)$ which could be used to compute this value. Consider the characteristic, $(s, \theta(s), M(s))$, passing through $(x_1, \theta_1, M(x_1, \theta_1))$, as $s \to \infty$. First, observe that for any finite interval $[s_1, s_2]$ the right side of the θ-equation is uniformly Lipschitzian in θ; hence the solution $\theta(s)$ can be extended indefinitely to the right. Moreover

$$M(s) = M(x_1)\exp\left\{-\int_{x_1}^{s}\sin\theta(p)\cos\theta(p)\,[f(p) - g(p)]\,dp\right\}$$

and, since the integrand is continuous, $M(s)$ can be extended indefinitely to the right. Finally, since $M(x, \theta) \to 1$ as $x \to \infty$ *uniformly in* θ it follows that

$$1 = \lim_{s\to\infty} M(s) = M(x_1)\exp\left\{-\int_{x_1}^{\infty}\sin\theta\cos\theta[f-g]\,dp\right\}.$$

This establishes the convergence of the infinite integral and provides the desired expression for $M(x_1, \theta_1)$;

$$M(x_1, \theta_1) = \exp\left\{\int_{x_1}^{\infty}\sin\theta(p)\cos\theta(p)\,[f(p) - g(p)]\,dp\right\},$$

where

$$\theta'(s) = \cos^2\theta(s)\,[f(s) - g(s)] - 1; \qquad \theta(x_1) = \theta_1.$$

In the same manner we obtain

$$\psi(x_1, \theta_1) = \int_{x_1}^{\infty}[\cos^2\theta(p)\,[f(p) - g(p)] - 1 + \sqrt{1 + g(p)}]\,dp.$$

THEOREM 3.1. *The solutions to problem (3.1) can be expressed*

$$u(t; x, \theta) = M(x, \theta) \cos \{\alpha(t) - \psi(x, \theta) - \theta - \alpha(x)\} + \xi(t; x, \theta),$$

where $\xi(t; x, \theta) \to 0$ *as* $t \to \infty$ *uniformly in* x *and* θ,

$$\alpha(t) = \int_b^t \sqrt{1+g}\, ds$$

and where $M(x, \theta)$ *and* $\psi(x, \theta)$ *are given by*

$$M(x, \theta) = \exp \left\{ \int_x^\infty \tfrac{1}{2} \sin 2\tilde{\theta}(f-g)\, ds \right\}$$

$$\psi(x, \theta) = \int_x^\infty [\cos^2 \tilde{\theta}(f-g) - 1 + \sqrt{1+g}]\, ds = \int_x^\infty [\tilde{\theta}'(s) + \sqrt{1+g}]\, ds$$

and $\tilde{\theta}$ *satisfies*

$$\theta'(s) = \cos^2 \theta(s)\, [f(s) - g(s)] - 1; \qquad \tilde{\theta}(x) = \theta.$$

We conclude this section with the following comments:

(1) If $1 + g(s) < 0$ for small values of s then $\sqrt{1+g(s)}$ is complex valued for small s; hence $\alpha(x)$ and $\psi(x, \theta)$ in our expression for $u(t; x, \theta)$ are complex valued. This can be avoided by writing

$$\psi(x, \theta) + \theta + \alpha(x) = \int_b^\infty [\tilde{\theta}'(s) + \sqrt{1+g}]\, ds + \int_x^b [\cdots]\, ds + \theta + \alpha(x)$$

$$= \int_b^\infty [\tilde{\theta}' + \sqrt{1+g}]\, ds + \tilde{\theta}(b)$$

and select b large enough that $1 + g(s) \geqslant 0$ for $s \geqslant b$.

(2) One can drop the requirement that $g(t) \to 0$ as $t \to \infty$ in (3.1c) and the above work is still valid. The complication is that $M(x, \theta)$, $\psi(x, \theta)$, and $\alpha(t)$ are then, in general, complex valued and would have to be treated as such in finding, for example, $|u(t; x, \theta)|$ as $t \to \infty$.

(3) If in addition to the assumption that $\int^\infty |g'| < \infty$ one requires $\int^\infty g^2 < \infty$ then it can be shown that the above expressions for $M(x, \theta)$ and $\psi(x, \theta)$ are the same except that one can replace $\sqrt{1+g}$ by $(1 + g/2)$ in $\alpha(t)$ and ψ and have

$$\alpha(t) = \int_b^t \left(1 + \frac{g}{2}\right) ds$$

$$\psi(x, \theta) = \int_x^\infty \left[\cos^2 \tilde{\theta}(f-g) - 1 + \left(1 + \frac{g}{2}\right)\right] ds$$

$$= \int_x^\infty [\cos^2 \tilde{\theta} f - \tfrac{1}{2} g \cos 2\tilde{\theta}]\, ds.$$

This has the advantage of not having to be concerned with complex values and the computational advantage of not having to take a square root.

IV. The Equation $u'' - (1 + g(t))\, u = f(t)\, u$

The second type of problem to be considered is

$$u'' - (1 + g(t))\, u = f(t)\, u, \qquad u(x) = \cos\theta, \qquad u'(x) = \sin\theta; \tag{4.1a}$$

$$\text{where } f \text{ is continuous and } \int_{x_0}^{\infty} |f| < \infty, \tag{4.1b}$$

and

$$g \text{ is absolutely continuous, } \int_{x_0}^{\infty} |g'| < \infty, \tag{4.1c}$$

$$\text{and } g(t) \to 0 \text{ as } t \to \infty.$$

For the equation $u'' - (1 + g)\, u = 0$ we are guaranteed two solutions such that

$$u_1(t) = e^{\alpha(t)}[k + o(1)], \qquad u_1'(t) = e^{\alpha(t)}[k + o(1)]$$

$$u_2(t) = e^{-\alpha(t)}[-k + o(1)], \qquad u_2'(t) = e^{-\alpha(t)}[k + o(1)] \tag{4.2}$$

as $t \to \infty$; where $\alpha(t) = \int_b^t \sqrt{1 + g}\, ds$ as before and k is an arbitrary constant. Selecting $k = 1/\sqrt{2}$ we have the Wronskian

$$[u_1(t), u_2(t)] = 1 + o(1) \equiv 1 \qquad \text{for all } t.$$

Rather than examining the solutions, $u(t; x, \theta)$, to (4.1) explicitly, it is convenient instead to study the function

$$z(t; x, \theta) \equiv e^{\alpha(x) - \alpha(t)} u(t; x, \theta). \tag{4.3}$$

We now show that z is uniformly bounded.

LEMMA 4.1. *For some $M < \infty$,*

$$| z(t; x, \theta) | \leqslant M$$

for $t \geqslant x \geqslant x_0$ and for all θ.

PROOF. Using (2.2) and (4.3) we can write

$$z(t; x, \theta) = e^{\alpha(x) - \alpha(t)} u_1(t) \left\{ \cos\theta u_2'(x) - \sin\theta u_2(x) - \int_x^t u_2 f z e^{\alpha(s) - \alpha(x)}\, ds \right\}$$

$$+ e^{\alpha(x) - \alpha(t)} u_2(t) \left\{ \sin\theta u_1(x) - \cos\theta u_1'(x) + \int_x^t u_1 f z e^{\alpha(s) - \alpha(x)}\, ds \right\}. \tag{4.4}$$

The result now follows from (4.2), (4.4) and the application of Gronwall's inequality as in Lemma 2.2.

Since z is bounded the convergence of the first integral in (4.4), as $t \to \infty$ is assured; so we can write

$$\begin{aligned} z(t; x, \theta) &= e^{\alpha(x)} \left(\frac{1}{\sqrt{2}} + o(1)\right) \\ &\times \left\{\cos \theta u_2'(x) - \sin \theta u_2(x) - \int_x^\infty [u_2 f z e^{\alpha(s)-\alpha(x)}]\, ds + \int_t^\infty [\cdots]\, ds\right\} \\ &- e^{\alpha(x)-2\alpha(t)} \left[\frac{1}{\sqrt{2}} + o(1)\right] \left\{\sin \theta u_1(x) - \cos \theta u_1'(x) + \int_x^t [u_1 f z e^{\alpha(s)-\alpha(x)}]\, ds\right\} \end{aligned} \tag{4.5}$$

as $t \to \infty$. We therefore define

$$A(x, \theta) = \frac{1}{\sqrt{2}} e^{\alpha(x)} \left\{\cos \theta u_2'(x) - \sin \theta u_2(x) - \int_x^\infty u_2 f z e^{\alpha(s)-\alpha(x)}\, ds\right\}. \tag{4.6}$$

Hence

$$z(t; x, \theta) = A(x, \theta) + \xi(t; x, \theta), \tag{4.7}$$

where $\xi \equiv z - A$, z defined by (4.4), and A by (4.6). It is easily shown that $\xi(t; x, \theta) \to 0$ as $t \to \infty$; hence $A(x, \theta)$ provides the asymptotic behavior of $z(t; x, \theta)$ and of $u(t; x, \theta)$. Finally, we note that

$$u(t; x, \theta) = e^{\alpha(t)-\alpha(x)}[A(x, \theta) + \xi(t; x, \theta)]$$

and we seek a computationally feasible expression for $A(x, \theta)$.

LEMMA 4.2. *For $A(x, \theta)$ defined by (4.6), A is continuously differentiable in x and θ for $x > x_0$.*

We omit the proof of this lemma which proceeds exactly like the proof of Lemma 2.4.

We now obtain a differential equation for A. Referring to Section II we know that

$$u(t; x, \theta) = \eta(x, \theta)\, u(t; x + \Delta, \theta'),$$

i.e.;

$$e^{\alpha(t)-\alpha(x)}[A(x, \theta) + \xi(t; x, \theta)] = \eta e^{\alpha(t)-\alpha(x+\Delta)}[A(x + \Delta, \theta') + \xi(t; x + \Delta, \theta')].$$

It follows that

$$\begin{aligned} e^{\alpha(x+\Delta)-\alpha(x)} A(x, \theta) &= \eta A(x + \Delta, \theta') \\ &= \eta \left\{A(x + \Delta, \theta) + (\theta' - \theta)\left[\frac{\partial A}{\partial \theta}(x, \theta) + o(1)\right]\right\} \end{aligned}$$

as $\Delta \to 0$. Using the fact that

$$e^{\alpha(x+\Delta)-\alpha(x)} = 1 + \Delta\sqrt{1+g(x)} + o(\Delta),$$

employing the expressions for η and θ' given in Lemma 2.3, and proceeding as in Section III, we get

$$\frac{\partial A}{\partial x} + \frac{\partial A}{\partial \theta}[\cos^2\theta(f+g+2) - 1] = A[\sqrt{1+g} - \sin\theta\cos\theta(f+g+2)]. \tag{4.8}$$

LEMMA 4.3. *$A(x, \theta)$ satisfies (4.8) and*

$$\lim_{x\to\infty}\left[A(x,\theta) - \frac{\cos\theta + \sin\theta}{2}\right] = 0 \quad \textit{uniformly in } \theta. \tag{4.9}$$

PROOF. It follows from Lemma 4.1 that the integral term in (4.6) goes to zero as $x \to \infty$ and uniformly so in θ. Finally

$$\frac{1}{\sqrt{2}}e^{\alpha(x)}[\cos\theta u_2'(x) - \sin\theta u_2(x)]$$

$$= \frac{1}{\sqrt{2}}\left[\cos\theta\left(\frac{1}{\sqrt{2}} + o(1)\right) - \sin\theta\left(-\frac{1}{\sqrt{2}} + o(1)\right)\right]$$

$$= \frac{\cos\theta + \sin\theta}{2} + o(1)$$

as $x \to \infty$ and the convergence is clearly uniform in θ. This establishes the lemma.

We now suggest two alternative ways for computing $A(x, \theta)$.

First, one may simply proceed as in Section III and apply the method of characteristics to (4.8). If this is done and (x_1, θ_1) is an arbitrary point, $x_1 \geqslant x_0$, then

$$A(x_1, \theta_1) = A(x_1)$$
$$= A(s)\exp\left\{\int_{x_1}^{s}[\sin\theta(p)\cos\theta(p)(f+g+2) - \sqrt{1+g}]\,dp\right\},$$

where

$$\theta'(s) = \cos^2\theta(s)(f+g+2) - 1; \qquad \theta(x_1) = \theta_1. \tag{4.10}$$

If $\lim_{s\to\infty} A(s) \equiv A(\infty)$ exists then our desired expression is

$$A(x_1, \theta_1) = A(\infty)\exp\left\{\int_{x_1}^{\infty}[\sin\theta\cos\theta(f+g+2) - \sqrt{1+g}]\,dp\right\}. \tag{4.11}$$

It is interesting to note that one is led to (4.10) when the Prüfer transformation is applied to (4.1a); see [8]. Using the fact that solutions to (4.1a) can be expressed as linear combinations of the forms in (4.2) and the fact that by [8]

$$\theta(s) = \arctan\left[\frac{u'(s)}{u(s)}\right],$$

we observe that

$$\theta(s) \to \arctan(\pm 1)$$

as $s \to \infty$. And therefore $\theta(s) \to (\pi/4) + m(\pi/2)$ as $s \to \infty$; where m is an even integer if $|u(s; x, \theta)| \to \infty$ as $s \to \infty$ and m is an odd integer if $u(s; x, \theta) \to 0$ as $s \to \infty$.

Using these facts we have the following theorem.

THEOREM 4.1. *The solutions to problem (4.1) can be expressed*

$$u(t; x, \theta) = e^{\alpha(t)-\alpha(x)}[A(x, \theta) + \xi(t; x, \theta)],$$

where $\xi(t; x, \theta) \to 0$ *as* $t \to \infty$,

$$\alpha(t) = \int_b^t \sqrt{1+g}\, ds$$

and

$$A(x, \theta) = \pm \frac{1}{\sqrt{2}} \exp\left\{\int_{x_1}^{\infty} [\sin\tilde{\theta}\cos\tilde{\theta}(f+g+2) - \sqrt{1+g}]\, ds\right\}, \tag{4.12}$$

where

$$\tilde{\theta}'(s) = \cos^2\tilde{\theta}(s)\,[f(s) + g(s) + 2] - 1; \qquad \tilde{\theta}(x) = \theta.$$

PROOF. By (4.9) we know

$$\lim_{s\to\infty}\left[A(s) - \frac{\cos\tilde{\theta}(s) + \sin\tilde{\theta}(s)}{2}\right] = 0$$

We have two cases to consider. First, if the integer m, defined above, is *even* then

$$A(\infty) = \frac{\cos\left[\frac{\pi}{4} + m\frac{\pi}{2}\right] + \sin\left[\frac{\pi}{4} + m\frac{\pi}{2}\right]}{2} = \mp\frac{1}{\sqrt{2}},$$

depending on whether $m/2$ is odd or even. Note that this also establishes the convergence of the infinite integral in (4.12).

On the other hand, if m is *odd* note that the integrand in (4.12) is

$\approx 2\sin\tilde{\theta}\cos\tilde{\theta} - 1 \approx -2$ for large s. So the integral in (4.12) is actually divergent (to $-\infty$). For this case we interpret (4.12) as

$$A(x, \theta) = \pm \frac{1}{\sqrt{2}} e^{-\infty} = 0$$

which is the correct result since $A(\infty) = 0$. This proves the theorem.

The algorithm suggested by Theorem 4.1 for computing $A(x, \theta)$ has some obvious disadvantages. In case $A(\infty) = 0$ and, hence $A(x, \theta) = 0$, one has a divergent integral to contend with in (4.12). Since in practice one does not know *a priori* that $A(\infty) = 0$ this presents a problem. If $A(x, \theta) \neq 0$, (4.12) does not provide the sign of $A(x, \theta)$. Both of these problems can be resolved by computing $\tilde{\theta}(s)$ over an interval of sufficient length to determine the limiting value of $\tilde{\theta}$. In fact Bailey [5] suggested solving the $\tilde{\theta}$-equation for large s as a way of determining the eigenvalues of the eigenvalue problem associated with (4.1a) (see Section VI below).

We now suggest an alternative way of computing $A(x, \theta)$ which takes care of the problems of the preceding paragraph and does so without determining the limiting value of $\tilde{\theta}(s)$. Define

$$B(x, \theta) = A(x, \theta) - \frac{\cos\theta + \sin\theta}{2}.$$

It is easily shown that the resulting boundary value problem is

$$\frac{\partial B}{\partial x} + \frac{\partial B}{\partial \theta}[\cos^2\theta(f + g + 2) - 1] = B[\sqrt{1+g} - \sin\theta\cos\theta(f + g + 2)] - \tfrac{1}{2}[(\sin\theta + \cos\theta)(1 - \sqrt{1+g}) + \cos\theta(f + g)] \tag{4.13}$$

$$\lim_{x\to\infty} B(x, \theta) = 0 \quad \textit{uniformly in } \theta. \tag{4.14}$$

This problem can also be solved by the method of characteristics. We state the result.

THEOREM 4.2. *The solutions to problem (4.1) can be expressed*

$$u(t; x, \theta) = e^{\alpha(t)-\alpha(x)}\left[B(x, \theta) + \frac{\sin\theta + \cos\theta}{2} + \xi(t; x, \theta)\right],$$

where

$$B(x, \theta) = \tfrac{1}{2}\int_x^\infty ds[(\sin\tilde{\theta} + \cos\tilde{\theta})(1 - \sqrt{1+g}) + (f + g)\cos\tilde{\theta}] \cdot \exp\left\{\int_x^s [\tfrac{1}{2}\sin 2\tilde{\theta}(f + g + 2) - \sqrt{1+g}]\, dp\right\}, \tag{4.15}$$

and where $\tilde{\theta}$ satisfies

$$\tilde{\theta}'(s) = \cos^2 \tilde{\theta}(f + g + 2) - 1; \qquad \tilde{\theta}(x) = \theta.$$

The result of Theorem 4.2 has the advantages mentioned above (in particular, it is easily shown that integral in (4.15) always converges) and the obvious disadvantage of a more complicated expression.

The remarks concluding Section III are in principle applicable here. Since later use will be made of the result we state the following fact in the form of the theorem.

THEOREM 4.3. *Let problem (4.1) have the additional hypothesis:*

$$\int_{x_0}^{\infty} g^2 < \infty;$$

then the solutions can be expressed

$$u(t; x, \theta) = e^{\alpha(t)-\alpha(x)} \left\{ B(x, \theta) + \frac{\sin\theta + \cos\theta}{2} + \xi(t; x, \theta) \right\}$$

where

$$\alpha(t) = \int_b^t \left(1 + \frac{g}{2}\right) ds$$

and

$$B(x, \theta) = \frac{1}{2} \int_x^{\infty} ds \left\{ f \cos\tilde{\theta} - \frac{g}{2} [\sin\tilde{\theta} - \cos\tilde{\theta}] \right\}$$

$$\cdot \exp \left\{ \int_x^s \left[\sin 2\tilde{\theta} \frac{f}{2} - \left(1 + \frac{g}{2}\right)(1 - \sin 2\tilde{\theta}) \right] dp \right\}$$

and

$$\tilde{\theta}'(s) = \cos^2 \tilde{\theta}(s) [f + g + 2] - 1, \qquad \tilde{\theta}(x) = \theta.$$

It is easily shown that the condition $\int^{\infty} g^2 < \infty$ allows one to write the solutions to $u'' - (1 + g) u = 0$ as (4.2) with $\alpha(t) \equiv \int_b^t (1 + g/2)\, ds$. The results of the theorem follow by proceeding exactly as before with this definition of $\alpha(t)$.

V. THE EQUATIONS $u'' = \tilde{f}(t)\, u$

The third and final general class of equations we consider is that which, roughly speaking, behaves like $u'' = 0$ for large t.

$$u'' = \tilde{f}(t)\, u, \qquad u(x) = \cos\theta, \qquad u'(x) = \sin\theta; \tag{5.1a}$$

$$\text{where } \tilde{f} \text{ is continuous and } \int_{x_0}^{\infty} t\, |\tilde{f}(t)|\, dt < \infty. \tag{5.1b}$$

Two independent solutions to $u'' = 0$ obviously are $u_1(t) = 1$ and $u_2(t) = t$.

Again applying (2.2) we have

$$u(t; x, \theta) = \left\{\cos\theta - x\sin\theta - \int_x^t s\tilde{f}u(s; x, \theta)\, ds\right\} + t\left\{\sin\theta + \int_x^t \tilde{f}u(s; x, \theta)\, ds\right\}. \tag{5.2}$$

We define

$$z(t; x, \theta) = \begin{cases} \dfrac{1}{t}\, u(t; x, \theta) & \text{for} \quad t \geqslant 1 \\ u(t; x, \theta) & \text{for} \quad t < 1 \end{cases} \tag{5.3}$$

and hence, for $t \geqslant 1$,

$$z(t; x, \theta) = \frac{1}{t}\left\{\cos\theta - x\sin\theta - \int_x^t s^2\tilde{f}z(s; x, \theta)\, ds\right\} + \left\{\sin\theta + \int_x^t s\tilde{f}z(s; x, \theta)\, ds\right\}.$$

As in the work of Section IV, it is easily shown that $z(t; x, \theta)$ is uniformly bounded for $t \geqslant x \geqslant x_0$ and for all θ. Moreover it follows that

$$\int_x^\infty s\tilde{f}z(s; x, \theta)\, ds \quad \text{exists}$$

and

$$\lim_{t\to\infty} \frac{1}{t}\int_x^t s^2\tilde{f}z(s; x, \theta)\, ds = 0.$$

We therefore write

$$\begin{aligned} z(t; x, \theta) &= \left[\sin\theta + \int_x^\infty s\tilde{f}z(s; x, \theta)\, ds\right] \\ &\quad + \frac{1}{t}\left[\cos\theta - x\sin\theta - \int_x^t s^2\tilde{f}z(s; x, \theta)\, ds\right] - \int_t^\infty s\tilde{f}z(s; x; \theta)\, ds \\ &\equiv \tilde{A}(x, \theta) + \xi(t; x, \theta) \end{aligned}$$

and it follows that $\xi(t; x, \theta) \to 0$ as $t \to \infty$.

Defining $\tilde{B}(x, \theta) = \tilde{A}(x, \theta) - \sin\theta$ and proceeding as before we find that $\tilde{B}(x, \theta)$ satisfies

$$\frac{\partial\tilde{B}}{\partial x} + [\cos^2\theta(\tilde{f} + 1) - 1]\frac{\partial\tilde{B}}{\partial\theta} = -\sin\theta\cos\theta(\tilde{f} + 1)\,\tilde{B} - \tilde{f}\cos\theta$$

$$\lim_{x\to\infty} \tilde{B}(x, \theta) = 0 \quad \textit{uniformly in } \theta. \tag{5.4}$$

We summarize the results in this theorem and omit the proof.

THEOREM 5.1. *The solutions to problem (5.1) can be expressed*

$$u(t; x, \theta) = t[\tilde{B}(x, \theta) + \sin\theta + \xi(t; x, \theta)]$$

where $\xi(t; x, \theta) \to 0$ *as* $t \to \infty$ *and*

$$\tilde{B}(x, \theta) = \int_{x_1}^{x} \tilde{f}(s) \cos\tilde{\theta}(s) \exp\left\{\int_x^s \tfrac{1}{2} \sin 2\tilde{\theta}(\tilde{f} + 1)\, dp\right\} ds,$$

where

$$\tilde{\theta}'(s) = \cos^2\tilde{\theta}(\tilde{f}(s) + 1) - 1, \qquad \tilde{\theta}(x) = \theta.$$

VI. SOME APPLICATIONS

In this section we illustrate how the results of the preceding sections can be used to provide computational algorithms for certain problems.

Some applications are obvious. For example, given a particular initial value problem obtained by specifying x and θ in (3.1), (4.1), or (5.1) one can clearly use the algorithm suggested in the corresponding theorem to compute an approximation (by truncating the infinite integral) to the desired coefficient.

Less apparent and perhaps more useful applications are provided by certain parameter studies associated with the initial value problem discussed. As our first example we discuss the following eigenvalue problem which has received a great deal of attention in the literature. Consider

$v''(s) - [F(s) + G(s)]\, v(s) = \mu v(s), \qquad v(0) = \cos\hat{\theta}, \qquad v'(0) = \sin\hat{\theta};$
where F and G satisfy the conditions on f and g, respectively, of Theorem 4.3 and $\hat{\theta}$ is specified. *Problem:* Find the real values of μ for which $v(s; \mu) \in L_2(0, \infty)$. (6.1)

We now discuss this problem in some detail. For $\mu \neq 0$ reference to Sections III and IV readily shows that $v(s; \mu) \in L_2(0, \infty)$ iff $v(s; \mu) \to 0$ as $s \to \infty$, so we will seek values of μ such that $v(s; \mu) \to 0$.

For $\mu \neq 0$ the differential equation in (6.1) can be put in standard form by the change in variables:

$$t \equiv \lambda s \equiv \sqrt{|\mu|}\, s; \qquad w(t) \equiv v(s), \tag{6.2}$$

and we obtain

$$w''(t) = \left[\frac{\mu}{|\mu|} + \frac{1}{|\mu|} G\left(\frac{t}{\lambda}\right)\right] w(t) = \frac{1}{|\mu|} F\left(\frac{t}{\lambda}\right) w(t). \tag{6.3}$$

If $\mu < 0$ the equation is of the type discussed in Section III. Recall that, as was pointed out in Section III, $w(t; \mu) \to 0$ as $t \to \infty$ only if $w(t; \mu) \equiv 0$.

It follows that $v(s; \mu)$ does *not* $\to 0$ as $s \to \infty$ and hence that *there are no negative eigenvalues* (which, of course, is a known fact).

We now consider (6.3) for $\mu > 0$. In order to apply the results of Section IV we first normalize the initial conditions for the Equation (6.3). Define

$$u(t) \equiv \frac{w(t)}{+\sqrt{\cos^2\hat{\theta} + \sin^2\hat{\theta}/\lambda^2}} \equiv \frac{w(t)}{k(\lambda)}$$

$$\cos\theta = \cos\theta(\lambda) \equiv \frac{\cos\hat{\theta}}{k(\lambda)}$$

$$\sin\theta = \sin\theta(\lambda) \equiv \frac{\sin\hat{\theta}}{\lambda k(\lambda)}.$$

It follows that $u^2(0) + [u'(0)]^2 = \cos^2\theta + \sin^2\theta = 1$. Also let

$$f(t;\lambda) \equiv \frac{F(t/\lambda)}{\lambda^2}$$

$$g(t;\lambda) \equiv \frac{G(t/\lambda)}{\lambda^2}$$

and the original initial value problem is *equivalent* to

$$u''(t) - [1 + g(t;\lambda)]\,u(t) = f(t;\lambda)\,u(t), \qquad u(0) = \cos\theta, \qquad u'(0) = \sin\theta; \tag{6.4}$$

i.e., clearly $v(s;\mu) \to 0$ as $s \to \infty$ iff $u(t;\lambda) \to 0$ as $t \to \infty$ where $\lambda = \sqrt{\mu}$. So we concentrate on (6.4) and describe a procedure for numerically locating values of λ such that $u(t;\lambda) \to 0$.

Using Theorem 4.3 we have

$$u(t;\lambda) = e^{\alpha(t;\lambda)-\alpha(0;\lambda)}\left\{B(\lambda) + \frac{\cos\theta + \sin\theta}{2} + \xi(t;\lambda)\right\}, \tag{6.5}$$

where

$$B(\lambda) = \frac{1}{2}\int_0^\infty \left\{f\cos\tilde{\theta} + \frac{g}{2}[\cos\tilde{\theta} - \sin\tilde{\theta}]\right\} \times \exp\left\{-\int_0^s \left[\left(1 + \frac{g}{2}\right)(1 - \sin 2\tilde{\theta}) - \frac{f}{2}\sin 2\tilde{\theta}\right] dp\right\} ds \tag{6.6}$$

with

$$\tilde{\theta}'(s) = \cos^2\tilde{\theta}(f + g + 2) - 1, \qquad \tilde{\theta}(0) = \theta = \theta(\lambda),$$

and

$$\alpha(t;\lambda) = \int_b^t [1 + \tfrac{1}{2}g(s;\lambda)]\,ds.$$

Note that the λ dependence in $B(\lambda)$ is concealed in f, g, and θ.

We now show that the eigenvalues of (6.2) are precisely the zeros of $A(\lambda)$ where

$$A(\lambda) \equiv B(\lambda) + \frac{\cos\theta(\lambda) + \sin\theta(\lambda)}{2}. \tag{6.7}$$

LEMMA 6.1. *A necessary and sufficient condition for $u(t; \lambda) \to 0$ as $t \to \infty$ is $A(\lambda) = 0$.*

PROOF. Clearly $e^{\alpha(t;\lambda)} \to \infty$ as $t \to \infty$. It follows from (6.5) that if $A(\lambda) \neq 0$ then $|u(t; \lambda)| \to \infty$ as $t \to \infty$ and the necessity condition is established.

Now assume $A(\lambda_1) = 0$. From standard theorems (e.g. see [3]) we are guaranteed two solutions to $u'' - (1 + g(t; \lambda_1))u = f(t; \lambda_1)u$ of the form

$$u_1(t; \lambda_1) = e^{\alpha(t;\lambda_1)}[1 + o(1)]$$
$$u_2(t; \lambda_1) = e^{-\alpha(t;\lambda_1)}[1 + o(1)]$$

as $t \to \infty$. Hence for some C_1 and C_2 the solution to (6.4) with $\lambda = \lambda_1$ can be written

$$u(t; \lambda_1) = C_1 e^{\alpha(t;\lambda_1)}[1 + o(1)] + C_2 e^{-\alpha(t;\lambda_1)}[1 + o(1)].$$

Since $A(\lambda_1) = 0$ it follows from (6.5) that

$$e^{-\alpha(t;\lambda_1)}u(t; \lambda_1) = e^{-\alpha(0;\lambda_1)}\xi(t; \lambda_1) \to 0$$

as $t \to \infty$. It follows that $C_1 = 0$ and hence that $u(t; \lambda_1) \to 0$ and the lemma is established.

Next we point out three rather important properties the $A(\lambda)$ possesses.

(1) When λ is considered as a complex variable $A(\lambda)$ is analytic for $\mathscr{R}e\lambda > 0$ (this can be shown by studying the expression for $A(\lambda)$ given in (4.6)). It follows from the identity theorem of complex analysis that the *zeros of $A(\lambda)$, and hence, the positive eigenvalues, are isolated.*

(2) It is easily shown that $B(\lambda) \to 0$ as $\lambda \to \infty$, hence

$$A(\lambda) \to \frac{\cos\theta(\lambda) + \sin\theta(\lambda)}{2} = \frac{1}{2}\,\frac{\lambda\cos\hat{\theta} + \sin\hat{\theta}}{\sqrt{\lambda^2\cos^2\hat{\theta} + \sin^2\hat{\theta}}} \to \pm\frac{1}{2}$$

as $\lambda \to \infty$.

Since $A(\lambda)$ is a continuous function of λ it follows that *the real eigenvalues are bounded.*

(3) Titchmarch [6] proved that, even under much more general conditions, *the real eigenvalues are simple.* It follows that the real zeros of $A(\lambda)$ are simple.

The fact that the real positive zeros are bounded, isolated, and simple is especially useful information if one wishes to use $A(\lambda)$ to locate the eigenvalues numerically. We now illustrate a simple procedure for doing this. Consider

$$v''(s) - \left[\frac{\beta(\beta+1)}{(s+1)^2} - \frac{1}{s+1}\right] v(s) = \lambda^2 v(s);$$
$$v(0) = \cos\hat{\theta}, \qquad v'(0) = \sin\hat{\theta}. \tag{6.8}$$

For particular values of the parameter β we wish to determine values of λ such that the solution $v(s; \lambda) \to 0$ as $s \to \infty$. For $\lambda = 1/2\beta$ this problem has closed form solutions, see Kamke [7], with which the following three test cases can easily be constructed:

(1) If $\beta = 1/2$ and $\theta = \tan^{-1}(-5/6)$, $v(s; 1) \to 0$

(2) If $\beta = 1$ and $\theta = \tan^{-1}(-7/10)$, $v(s; 1/2) \to 0$

(3) If $\beta = 3/2$ and $\theta = \tan^{-1}(-3345/2826)$, $v(s; 1/3) \to 0$.

We calculated λ approximately for each of these cases as follows:

(1) $A(\lambda)$ was computed for a discrete set of λ; $\lambda_1, \ldots, \lambda_n$. The sign changes between successive values of A indicate approximate locations of eigenvalues.

(2) using the approximate values given by step (1), the method of false position was applied until the desired eigenvalues were given to several decimal places.

The numerical results are shown on the following tables. Note that Table 6.1 indicates the existence of several eigenvalues whose value we did not pursue further.

Two questions concerning the original problem, (6.1), we have not answered. First, it is possible that $\mu = 0$ is an eigenvalue and we are in position to determine this only if $F(s)$ and $G(s)$ are small enough for large s to allow the results of Section V to be applied. Secondly, it is possible that $\mu = 0$ is the limit point of an infinite sequence of positive eigenvalues. If this is the case, clearly the simple procedure just outlined would have to be refined in order to provide a good method for computing a large number of small eigenvalues.

We feel that the properties of the function $A(\lambda)$ make it useful in the type of application discussed above. Probably the main feature of this approach, in studying the above eigenvalue problem, is that it allows one to study a highly unstable problem with an algorithm which apparently has good convergence properties (problem (6.1) is highly unstable in the usual sense due to the exponential behavior of the solutions when $\mu > 0$). In particular,

TABLE 6.1

λ	Problem 1 $\tan\theta = -5/6$ $\lambda = 1$ is an eigenvalue $A(\lambda)$	Problem 2 $\tan\theta = -7/10$ $\lambda = \frac{1}{2}$ is an eigenvalue $A(\lambda)$	Problem 3 $\tan\theta = -3345/2826$ $\lambda = \frac{1}{3}$ is an eigenvalue $A(\lambda)$
.15	−134.69921	−23.97329	38.17122
.25	1.49441	−1.77302	−1.48636
.35	−1.20597	−.93510	.12226
.45	−.93347	−.18989	.44316
.55	−.61865	.12811	.52083
.65	−.39547	.28208	.54398
.75	−.23802	.36587	.55188
.85	−.12297	.41544	
.95	−.03553	.44652	
1.05	.03056	.46682	
1.15	.08590	.48046	⋮
1.25	.12987	.48981	
⋮	⋮	⋮	
3.0	.37996	.50683	.52706

TABLE 6.2

starting interval is [.95, 1.05]		starting interval is [.45, .55]		starting interval is [.25, .35]	
λ	$A(\lambda)$	λ	$A(\lambda)$	λ	$A(\lambda)$
1.00376	.00249	.50971	.02664	.33775	.03154
1.00024	.00020	.50237	.00685	.33483	.01138
.99996	.00002	.50054	.00164	.33381	.00373
		.50011	.00038	.33347	.00117
		.50001	.00009	.33337	.00036
		.49999	.00002	.33334	.00011

computing experience has demonstrated the fortunate tendency of the algorithm for computing $A(\lambda)$ to converge more and more rapidly as λ approaches an eigenvalue. This behavior can be explained by careful analysis of the expressions in Theorem 4.3.

We close this section by suggesting another type of application which seems rather natural. Consider the following very simple control problem

$$u'' + (1 + g(t))\, u = f(t; \rho)\, u; \qquad u(x) = \cos\theta, \qquad u'(x) = \sin\theta;$$

where g and f (for fixed ρ) are as in (3.1) and x and θ are specified. *Problem:* find the value(s) of ρ, $\rho_1 \leqslant \rho \leqslant \rho_2$, such that $u(t; \rho)$ has minimum amplitude as $t \to \infty$.

One can apply Theorem 3.1 and compute amplitude $M = M(\rho)$ for a discrete set of ρ-values and thereby approximate the optimal value(s) of ρ. Quite possibly more imaginative applications of Theorem 3.1 can be found; but we do not pursue this problem here.

VII. Possible Generalizations

The problems treated in Sections III-IV all involve linear second-order differential equations which can be considered as "perturbations" of constant coefficient equations. We now briefly discuss the possibility of applying the technique to other types of problems.

Recall that in Section II we initiated the method for the more general equation

$$u'' + c(t)\,u = f(t)\,u, \tag{7.1}$$

where $\int^{\infty} |f| < \infty$. However, in order to complete the work, as in the following sections, more specific behavior for $c(t)$ was required so that the asymptotic behavior of two solutions to $u'' + c(t)\,u = 0$ was known. It seems very unlikely that problems involving the equation (7.1) can be handled in great generality. However in the case of *specific* equations of the type (7.1), but not having one of the three standard forms, we can offer two suggestions:

(1) Various transformations are available which, on occasions, transform the origin equation into a standard form; for example, see [3].

(2) If the asymptotic behavior of two independent solutions to $u'' + c(t)\,u = 0$ are known then it is very possible that one can successfully carry the work to completion. For example, the author considered the equation

$$u'' - (k^2t^2 + k)\,u = f(t)\,u, \tag{7.2}$$

where $k > 0$. For $f \equiv 0$ this equation can be solved; two independent solutions are

$$u_1(t) = e^{\alpha^*(t)}$$

$$u_2(t) = 2ke^{\alpha^*(t)} \int_t^{\infty} e^{-2\alpha^*(s)}\,ds = t^{-1}e^{-\alpha^*(t)}[1 + o(1)]$$

as $t \to \infty$, where $\alpha^*(t) = kt^2/2$. Proceeding as in Section IV we expressed solutions to (7.2) as $u(t; x, \theta) = e^{\alpha^*(t)}[A^*(x, \theta) + \xi(t; x, \theta)]$ and established the results analogous Theorems 4.1 and 4.2.

A possible generalization is to nonlinear perturbation. For example,

$$u'' \pm (1 + g(t))\,u = f(t, u), \qquad u(x) = a, \qquad u'(x) = b, \tag{7.3}$$

where $|f(t, u)| \leqslant g(t) | u(t) |$ and $\int^{\infty} g < \infty$. This problem can be studied using the imbedding technique; however, the nonlinearity is costly and the results, at this time, are probably not of practical interest. We feel that this problem merits more study.

Finally, it is natural to consider higher order equations. Work is currently being done in this area. Higher order equations complicate the problem considerably and some trouble spots still exist but it is hoped that some interesting results can be achieved.

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